Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 5, 2003*
Accepted: *October 28, 2003*

Holography and hydrodynamics: diffusion on stretched horizons

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JHEP10(2003)064

Abstract: We show that long-time, long-distance fluctuations of plane-symmetric horizons exhibit universal hydrodynamic behavior. By considering classical fluctuations around black-brane backgrounds, we find both diffusive and shear modes. The diffusion constant and the shear viscosity are given by simple formulas, in terms of metric components. For a given metric, the answers can be interpreted as corresponding kinetic coefficients in the holographically dual theory. For the near-extremal Dp, M2 and M5 branes, the computed kinetic coefficients coincide with the results of independent AdS/CFT calculations. In all the examples, the ratio of shear viscosity to entropy density is equal to $\hbar/(4\pi k_B)$, suggesting a special meaning of this value.

Keywords: p-branes, Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence, Black Holes.

Contents

JHEP10(2003)064

1. Introduction

The holographic principle [1, 2, 3] is believed to be fundamental to the construction of the quantum theory of gravity. It asserts that the physics in a region of space, in a theory with gravity, can be described by a set of degrees of freedom (some "dual theory", which does not contain gravity), associated with a hypersurface of smaller dimension in that space (the "holographic screen"). The nature of the degrees of freedom on the holographic screen is not specified by the holographic principle. In the explicit example of holography discovered in string theory — the AdS/CFT duality [4]–[7], the holographic degrees of freedom form a local quantum field theory ($\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions), but in general this need not be the case.

In this paper we elucidate the infrared properties of theories whose gravity duals contain a black brane with a nonzero Hawking temperature. We argue that the infrared behavior of these theories is governed by nothing other than *hydrodynamics*. Namely, by considering fluctuations around generic black brane solutions we found modes with dispersion relations of the type $\omega = -iDq^2$, which are naturally interpreted as diffusion of conserved charges and damped hydrodynamic shear flow in the holographically dual theory.[1]

This result can be anticipated from the holographic principle. Indeed, the dual theory is at a finite temperature, equal to the Hawking temperature associated with the gravitational background, and is translationally invariant. On general grounds, we expect the infrared behavior of such a theory to be governed by hydrodynamics. The existence of the diffusive modes is consistent with this expectation.

Some hydrodynamic-like properties of event horizons have been known for a long time in the framework of the "membrane paradigm" [8, 9]. In this approach the stretched horizon is interpreted as a fluid with certain dissipative properties such as electrical conductivity and shear and bulk viscosities. Normally, the "membrane paradigm" is applied to black holes. However, for black holes the relation to hydrodynamics does not go beyond the level of analogy, in particular because their horizons lack translational invariance. The connection to hydrodynamics is much more direct in the case of black branes. The membrane paradigm suggests the mapping between the bulk fields and the currents, whereas the long wavelength limit has to be taken to derive the hydrodynamic equations.

Using the AdS/CFT correspondence, it has been shown [10]–[14] that the theories living on non-extremal D3, M2 and M5 branes behave hydrodynamically in the infrared. In this paper, we will see hydrodynamic behavior emerging as a common feature of black brane backgrounds. Our calculations do not rely on a particular realization of holography such as AdS/CFT. We find explicit formulas expressing the transport coefficients (diffusion rate, shear viscosity) in terms of the components of the metric for a wide class of metrics.

The paper is organized as follows. In section 2 we consider small fluctuations of black brane backgrounds, and demonstrate the emergence of Fick's law and diffusion from Maxwell's equations. We derive an explicit formula for the diffusion constant. In section 3 we extend the discussion to gravitational perturbations and compute the shear viscosity. In section 4 the derived formulas are applied to various supergravity backgrounds. In the D3, M2, and M5 examples, the answers coincide with the kinetic coefficients in the dual theories, computed previously by AdS/CFT methods. In section 5 the kinetic coefficients in the theories dual to supergravity on Dp brane backgrounds are computed using the AdS/CFT recipe. The results coincide with those obtained in section 4. Section 6 discusses some unsolved problems and possible directions for future work.

[1] The gravity dual of the sound wave, with the dispersion relation $\omega = v_s q - i\gamma q^2$, should also exist, but is beyond the scope of this paper.

2. Diffusive and shear modes from horizon fluctuations

2.1 Backgrounds

We start by considering linearized perturbations of a black-brane background. We take the D-dimensional background metric G_{MN} to be of the form[2]

$$ds^2 = G_{00}(r)\,dt^2 + G_{rr}(r)\,dr^2 + G_{xx}(r)\sum_{i=1}^{p}(dx^i)^2 + \,Z(r)\,K_{mn}(y)\,dy^m dy^n\,, \tag{2.1}$$

where the components $G_{00}(r)$, $G_{rr}(r)$, $G_{xx}(r)$, and the "warping factor" $Z(r)$ depend only on the radial coordinate r.[3] We assume that the metric has a plane-symmetric horizon at $r = r_0$, that extends in p infinite spatial dimensions parametrized by the coordinates x^i. In the vicinity of the horizon G_{00} vanishes, G_{rr} diverges, and G_{xx}, Z stay finite. The coordinates y^m parametrize some d-dimensional compact space.

We will be interested in small fluctuations of the black brane background, $G_{MN}+\tilde{G}_{MN}$, where $\tilde{G}_{MN}$ is a small perturbation, and indices M, N run from 0 to $(1+p+d)$. As usual, such perturbations can be divided up into "scalar", "vector", and "tensor" parts: $\tilde{G}_{mn}$, $\tilde{G}_{\mu n}$, and $\tilde{G}_{\mu\nu}$, correspondingly. Here indices μ, ν run from 0 to $p+1$ (labeling the "non-compact" coordinates t, x^i and r), and indices m, n run from 1 to d (labeling the "compact" coordinates y^m). For our purposes we will be interested in vector and tensor perturbations only; this is motivated by the fact that effective hydrodynamics in field theories is constructed in terms of conserved currents and energy-momentum tensor of the theory.

As in the Kaluza-Klein mechanism, the problem of analyzing y-independent tensor/vector perturbations reduces to the problem of gravitational/vector fields propagating on a lower-dimensional background. We write the metric of the dimensionally reduced background as

$$ds^2 = g_{00}(r)\,dt^2 + g_{rr}(r)\,dr^2 + g_{xx}(r)\sum_{i=1}^{p}(dx^i)^2\,. \tag{2.2}$$

Its components $g_{\mu\nu}$ are just $G_{\mu\nu}$ of (2.1), multiplied by $Z(r)^{d/p}$. (Details of dimensional reduction are sketched in appendix A.)

We will restrict ourselves to the linearized equations of motion for $\tilde{G}_{\mu\nu}$ and $\tilde{G}_{\mu n}$ on the background (2.2). The dynamics of the vector perturbations is governed by Maxwell's action

$$S_{\text{gauge}} \sim \int d^{p+2}x\,\sqrt{-g}\,\left(\frac{1}{g_{\text{eff}}^2}\,F^{\mu\nu}F_{\mu\nu}\right), \tag{2.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with the gauge field A_μ being proportional to $\tilde{G}_{\mu n}$. In (2.3),

[2]Black brane metrics are solutions to the low-energy string- or M-theory (ten or eleven-dimensional supergravity) equations of motion. Full solutions also include other tensor fields such as the Ramond-Ramond field and/or the dilaton. However, to the linear order in gravitational perturbations, the presence of these extra fields does not affect subsequent discussion.

[3]Our choice of metric excludes rotating branes. In other words, we do not consider nonzero chemical potentials.

$g_{\rm eff}$ is an effective coupling, whose overall normalization will not be important. If $Z(r)$ is not constant (the compact space does not factorize), then $g^2_{\rm eff}$ is r-dependent, and is proportional to $Z(r)^{-(p+d)/p}$.

The dimensionally reduced metric (2.2) has an event horizon at $r = r_0$, in the vicinity of which g_{00} vanishes and g_{rr} diverges,

$$g_{00} = -(r - r_0)\,\gamma_0 + O((r - r_0)^2)\,, \tag{2.4a}$$

$$g_{rr} = \frac{\gamma_r}{r - r_0} + O(1)\,. \tag{2.4b}$$

Here γ_0 and γ_r are some positive constants. We assume that $g_{xx}(r)$ is a slowly varying function of r which does not vanish or diverge at the horizon, and is of the same order as g_{00} when $r - r_0 \gtrsim r_0$. The Hawking temperature associated with the dimensionally reduced background (2.2) is equal to the Hawking temperature of the full metric (2.1), and is given by

$$T = \frac{1}{4\pi}\sqrt{\frac{\gamma_0}{\gamma_r}}\,. \tag{2.5}$$

We start by analyzing vector perturbations. Tensor perturbations $\tilde{G}_{\mu\nu}$ will be considered in the next section.

2.2 Current

We shall illustrate the appearance of hydrodynamics on a simple problem of an abelian gauge field on the black-brane background (2.2). Maxwell's equations, which follow from the action (2.3), read

$$\partial_\mu\left(\frac{1}{g^2_{\rm eff}}\sqrt{-g}\,F^{\mu\nu}\right) = 0\,. \tag{2.6}$$

To simplify the expressions, we will take $g_{\rm eff}$ to be constant; thus Maxwell's equations become $\partial_\mu\left(\sqrt{-g}\,F^{\mu\nu}\right) = 0$. The results for the r-dependent $g_{\rm eff}$ can be easily restored by replacing $\sqrt{-g} \to \sqrt{-g}/g^2_{\rm eff}$ in the final answers.

We will be considering fluctuations of the gauge field, which behave as $A_\mu \propto e^{-i\omega t + i\boldsymbol{q}\cdot\boldsymbol{x}}$. Equations of motion for the field A_μ, together with appropriate boundary conditions give the dispersion law $\omega = \omega(q)$. In the limit $\boldsymbol{q} \to 0$, the dispersion law of the form $\omega(q) = -iD\boldsymbol{q}^2$, with D constant, is a sign of diffusion. We will be looking for the diffusive dispersion law on the black brane background (2.2).

In principle, one can derive the dispersion law by directly solving Maxwell's equations. Here we will follow a somewhat different path. A conserved "current" j^μ can be defined directly in terms of the field strength $F_{\mu\nu}$. One can show that if the gauge field satisfies the equations of motion with relevant boundary conditions, then at long distances Fick's law $j^i = -D\,\partial_i j^0$ is valid. Then current conservation $\partial_\mu j^\mu = 0$ implies that the charge density j^0 satisfies the diffusion equation $\partial_t\, j^0 = D\,\partial_i\partial_i\, j^0$.

To define the current we need to introduce the *stretched horizon* [8]. In our case, the stretched horizon is just a flat spacelike surface located at a constant $r = r_{\rm h}$ slightly larger than r_0:

$$r_{\rm h} > r_0\,, \qquad r_{\rm h} - r_0 \ll r_0\,. \tag{2.7}$$

The outward normal to that surface is a spacelike vector n_μ, which has only one nonzero component $n_r = g_{rr}^{1/2}(r_\mathrm{h})$, so that $n^2 = g^{rr} n_r n_r = 1$. The current which is associated with the stretched horizon is defined as in the original membrane paradigm approach [8, 9]:

$$j^\mu = n_\nu F^{\mu\nu}|_{r_\mathrm{h}} \,. \tag{2.8}$$

The antisymmetry of $F^{\mu\nu}$ implies $n_\mu j^\mu = 0$. That is, the current is parallel to the horizon. Further, Maxwell's equations, and the fact that all metric components $g_{\mu\nu}$ depend on r only, imply that j^μ is conserved, $\partial_\mu j^\mu = 0$, for any choice of r_h. The components of the current in the vicinity of the horizon are

$$j^0 = F^{0r} n_r = -\frac{F_{0r}}{\gamma_0 \gamma_r^{1/2} (r_\mathrm{h} - r_0)^{1/2}} \,, \tag{2.9a}$$

$$j^i = F^{ir} n_r = \frac{(r_\mathrm{h} - r_0)^{1/2}}{g_{xx} \gamma_r^{1/2}} F_{ir} \,. \tag{2.9b}$$

The diffusion equation can be derived if we can prove Fick's law. We shall do that in two steps:

- From the incoming-wave boundary conditions we infer a relation between electric and magnetic fields near the boundary:

$$F_{ir} = -\sqrt{\frac{\gamma_r}{\gamma_0}} \frac{F_{0i}}{r - r_0} \,, \qquad r - r_0 \ll r_0 \,. \tag{2.10}$$

  This relation is analogous to the relation $\mathbf{B} = -\mathbf{n} \times \mathbf{E}$ for plane waves on a non-reflecting surface in classical electrodynamics.

- We then show that if A_μ varies slowly in space and time, then F_{0i} is proportional to $\partial_i F_{0r}$ (analogously to $\mathbf{E} = -\boldsymbol{\nabla}\varphi$ in electrostatics), for appropriate choices of r_h. Then the relations (2.9a), (2.9b) imply $j^i = -D\, \partial_i j^0$, and give the expression for D.

We now present details of the derivation.

2.3 Boundary conditions

Because of translation invariance of the metric (2.2), we can restrict ourselves to field configurations which are plane waves with respect to the spatial coordinates x^i. Without loosing generality, we choose the wavevector to lie along the $x \equiv x^1$ axis, so

$$A_\mu = A_\mu(t, r) e^{iqx} \,. \tag{2.11}$$

In the hydrodynamics limit q is arbitrarily small. In particular we will assume it is much smaller than the Hawking temperature: $q \ll T$.

Fick's law $j^i = -D\, \partial_i j^0$ violates time reversal. Its origin thus must be rooted in the irreversible nature of the horizon. To incorporate time irreversibility we impose the incoming-wave boundary conditions on the horizon: waves can be absorbed by the horizon but cannot be emitted from there.

To derive the relation between electric and magnetic fields, we first write down the relevant Maxwell's equations and Bianchi identities, taking into account the form (2.11)

$$
\begin{aligned}
g^{00}\partial_t F_{0r} - g^{xx}\,\partial_x F_{rx} &= 0\,, &(2.12\text{a})\\
\sqrt{-g}\,g^{00} g^{xx}\partial_t F_{0x} + \partial_r(\sqrt{-g}\,g^{rr} g^{xx} F_{rx}) &= 0\,, &(2.12\text{b})\\
\partial_r(\sqrt{-g}\,g^{rr} g^{00} F_{0r}) + \sqrt{-g}\,g^{xx} g^{00}\,\partial_x F_{0x} &= 0\,, &(2.12\text{c})\\
\partial_t F_{rx} + \partial_x F_{0r} - \partial_r F_{0x} &= 0\,. &(2.12\text{d})
\end{aligned}
$$

Using eq. (2.12d) one can write down field equations containing only the electric fields F_{0r} and F_{0x},

$$
\begin{aligned}
\partial_t^2 F_{0r} - g_{00}\,g^{xx} q^2 F_{0r} - (iq) g_{00}\,g^{xx}\partial_r F_{0x} &= 0\,, &(2.13\text{a})\\
\partial_t^2 F_{0x} + g_{00}\,\partial_r(g^{rr}\partial_r F_{0x}) - g_{00}\,\partial_r(g^{rr}\,\partial_x F_{0r}) &= 0\,. &(2.13\text{b})
\end{aligned}
$$

In the near-horizon region $r - r_0 \ll r_0$, these equations simplify considerably. Let us assume that fields vary over a typical time scale Γ^{-1}, so that $\partial_t^2 \sim \Gamma^2$. At small $(r - r_0)$ eq. (2.13a) implies

$$F_{0r} \sim \frac{r - r_0}{r_0}\frac{q}{\Gamma^2}\partial_r F_{0x}\,. \tag{2.14}$$

Hence at sufficiently small $r - r_0$, namely, when

$$\frac{r - r_0}{r_0} \ll \frac{\Gamma^2}{q^2}\,, \tag{2.15}$$

the third term in the left hand side of eq. (2.13b) is small compared to the second term. This equation gives a "wave equation" for F_{0x} alone,

$$\partial_t^2 F_{0x} - \frac{\gamma_0}{\gamma_r}(r - r_0)\,\partial_r[(r - r_0)\partial_r F_{0x}] = 0\,. \tag{2.16}$$

The general solution to that equation is

$$F_{0x}(t,r) = f_1\left[t + \sqrt{\frac{\gamma_r}{\gamma_0}}\ln(r - r_0)\right] + f_2\left[t - \sqrt{\frac{\gamma_r}{\gamma_0}}\ln(r - r_0)\right], \tag{2.17}$$

where f_1 and f_2 are arbitrary functions. The incoming-wave boundary condition picks up the f_1 term. This means

$$\partial_t F_{0x} = \sqrt{\frac{\gamma_0}{\gamma_r}}(r - r_0)\partial_r F_{0x}\,. \tag{2.18}$$

From eq. (2.12d) and eq. (2.14) one finds that

$$F_{rx} - \sqrt{\frac{\gamma_r}{\gamma_0}}\frac{F_{0x}}{r - r_0} \tag{2.19}$$

is independent of t. As we expect the solution to decay as $t \to \infty$, this expression is zero. Thus we find eq. (2.10).

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2.4 Quick derivation of Fick's law

From now on we will work in the radial gauge $A_r = 0$. We shall show that the parallel (to the horizon) electric field is dominated by the gradient of the scalar potential, $F_{0x} \approx -\partial_x A_0$, and that the normal (to the horizon) electric field is proportional to the scalar potential itself, $A_0 \sim F_{0r}$. Hence one finds $F_{0x} \sim \partial_x F_{0r}$, and therefore $j^x \sim \partial_x j^0$.

When $q = 0$, A_0 satisfies Poisson equation (see eq. (2.12c))

$$\partial_r \left(\sqrt{-g}\, g^{rr} g^{00}\, \partial_r A_0\right) = 0\,. \tag{2.20}$$

With the boundary condition $A_0(r) = 0$ at $r \to \infty$, it can be solved to yield

$$A_0(r) = C_0 \int\limits_r^\infty dr'\, \frac{g_{00}(r')\, g_{rr}(r')}{\sqrt{-g(r')}}\,. \tag{2.21}$$

Now let q be nonzero but small. Our first assumption, (to be checked later), is that unless r is exponentially close to r_0 one can expand A_0 in a series over $q^2/T^2 \ll 1$, and the leading term has the same r-dependence as in eq. (2.21), with C_0 depending on coordinates and time, i.e.,

$$\begin{aligned} A_0 &= A_0^{(0)} + A_0^{(1)} + \cdots\,, \quad A_0^{(1)} = O\left(\frac{q^2}{T^2}\right)\,, \\ A_0^{(0)}(t,x,r) &= C_0(t) e^{iqx} \int\limits_r^\infty dr'\, \frac{g_{00}(r')\, g_{rr}(r')}{\sqrt{-g(r')}}\,, \end{aligned} \tag{2.22}$$

When $r \approx r_0$ (but not exponentially close to r_0) the ratio of F_{0r} and A_0 is a constant which can be read off from the metric,

$$\left.\frac{A_0}{F_{0r}}\right|_{r\approx r_0} = \frac{\sqrt{-g(r_0)}}{g_{00}(r_0)\, g_{rr}(r_0)} \int\limits_{r_0}^\infty dr\, \frac{g_{00}(r) g_{rr}(r)}{\sqrt{-g(r)}}\,. \tag{2.23}$$

Note that this ratio is finite since $g_{00}\, g_{rr}$ is finite as $r \to r_0$. Our second assumption is that

$$|\partial_t A_x| \ll |\partial_x A_0|\,. \tag{2.24}$$

Now Fick's law arises naturally:

$$\begin{aligned} j^x &= -\frac{F_{0x}}{g_{xx}\sqrt{\gamma_0(r - r_0)}} \\ &= \frac{\partial_x A_0}{g_{xx}\sqrt{\gamma_0(r-r_0)}} \\ &= \left(\frac{A_0}{F_{0r}}\right) \frac{\partial_x F_{0r}}{g_{xx}\sqrt{\gamma_0(r-r_0)}} \\ &= -D\partial_x j^0\,, \end{aligned} \tag{2.25}$$

where

$$D = \frac{\sqrt{-g(r_0)}}{g_{xx}(r_0)\sqrt{-g_{00}(r_0)\,g_{rr}(r_0)}} \int\limits_{r_0}^{\infty} dr\, \frac{-g_{00}(r)\,g_{rr}(r)}{\sqrt{-g(r)}}\,. \tag{2.26}$$

For the gauge field with r-dependent coupling $g_{\rm eff}(r)$, the diffusion constant is

$$D = \frac{\sqrt{-g(r_0)}}{g_{xx}(r_0)\,g_{\rm eff}^2(r_0)\sqrt{-g_{00}(r_0)\,g_{rr}(r_0)}} \int\limits_{r_0}^{\infty} dr\, \frac{-g_{00}(r)\,g_{rr}(r)\,g_{\rm eff}^2(r)}{\sqrt{-g(r)}}\,. \tag{2.27}$$

The natural estimate for the diffusion constant is $D \sim T^{-1}$. From Fick's law and continuity it follows that j^0 obeys the diffusion equation, and $C_0(t) \propto e^{-\Gamma t}$, $\Gamma = Dq^2$. Notice that (2.15) requires that the stretched horizon has to be sufficiently close to the horizon,

$$\frac{r_{\rm h} - r_0}{r_0} \ll \frac{q^2}{T^2}\,, \tag{2.28}$$

which is a stronger condition than $r_{\rm h} - r_0 \ll r_0$.

2.5 Checking the assumptions

The short derivation of the diffusion law relies on two assumptions, eqs. (2.22) and (2.24). We now verify that these assumptions are self-consistent. Namely, we assume eq. (2.22) and then show that i) (2.24) is valid and ii) (2.22) satisfies field equations.

From eq. (2.12a)

$$-\Gamma g^{00} A_0' + iqg^{xx} A_x' = 0 \tag{2.29}$$

(where prime denotes derivative with respect to r) one can find A_x from A_0. Combining that with eq. (2.22) and taking into account the boundary condition $A_x|_{r=\infty} = 0$, one finds

$$\begin{aligned} A_x &= A_x^{(0)} + A_x^{(1)} + \cdots\,, \\ A_x^{(0)} &= -\frac{i\Gamma}{q} C_0 e^{-\Gamma t + iqx} \int\limits_{r}^{\infty} dr'\, \frac{g_{xx}(r') g_{rr}(r')}{\sqrt{-g(r')}}\,. \end{aligned} \tag{2.30}$$

For $r - r_0 \sim r_0$ we have $A_x/A_0 \sim \Gamma/q$. This comes from comparing eqs. (2.30) and (2.22), taking into account that $g_{00} \sim g_{xx}$ for these values of r. However in the limit $r \to r_0$ the integral diverges logarithmically since $g_{rr} \sim (r - r_0)^{-1}$. For very small $r - r_0$ we then have

$$A_x \sim A_0 \frac{\Gamma}{q} \ln \frac{r_0}{r - r_0}\,. \tag{2.31}$$

Let us look at the condition (2.24). We have

$$\frac{|\partial_t A_x|}{|\partial_x A_0|} \sim \frac{\Gamma^2}{q^2} \ln \frac{r_0}{r - r_0}\,. \tag{2.32}$$

Since $\Gamma^2/q^2 \sim q^2/T^2 \ll 1$, the ratio is much smaller than 1 unless $r - r_0$ is exponentially small so that the logarithm is comparable to T^2/q^2. Therefore (2.24) holds if we choose the location of the stretched horizon r_h so that

$$\ln \frac{r_0}{r_\mathrm{h} - r_0} \ll \frac{T^2}{q^2} \,. \tag{2.33}$$

This means $r_h - r_0$ cannot be too small. Still this condition can be satisfied simultaneously with (2.28).

Now let us check that A_0 can be expanded as in eq. (2.22). Expanding eq. (2.12c) in series over q^2 we find

$$\partial_r \left(\sqrt{-g} g^{rr} g^{00} \partial_r A_0^{(0)} \right) + \partial_x \left[\sqrt{-g} g^{xx} g^{00} \left(\partial_x A_0^{(0)} - \partial_t A_x^{(0)} \right) \right] = 0 \,. \tag{2.34}$$

Concentrating on r close to r_0, and only on the orders of magnitude of the terms, we find

$$\partial_r^2 A_0^{(1)} \sim \frac{\gamma_r g^{xx}}{r - r_0} \left(q^2 A_0^{(0)} + \Gamma q A_x^{(1)} \right) \sim \frac{\gamma_r g^{xx}}{r - r_0} \left(q^2 + \Gamma^2 \ln \frac{r_0}{r - r_0} \right) A_0^{(0)} \,, \tag{2.35}$$

from which it follows that

$$\begin{aligned} \partial_r A_0^{(1)} &\sim \gamma_r g^{xx} \left(q^2 \ln \frac{r_0}{r - r_0} + \Gamma^2 \ln^2 \frac{r_0}{r - r_0} \right) A_0^{(0)} \\ &\sim \frac{1}{r_0} \left(\frac{q^2}{T^2} \ln \frac{r_0}{r - r_0} + \frac{q^4}{T^4} \ln^2 \frac{r_0}{r - r_0} \right) A_0^{(0)} \,, \end{aligned} \tag{2.36}$$

where we have used the estimate $g_{xx} \sim \gamma_0 r_0$ and $\Gamma \sim q^2/T$. We see that $A_0^{(1)}$ is always smaller than $A_0^{(0)}$ for small q^2/T^2. However, using eq. (2.29) one finds

$$A_x^{(1)} \sim \frac{\Gamma}{q} \left(\frac{q^2}{T^2} \ln^2 \frac{r_0}{r - r_0} + \frac{q^4}{T^4} \ln^3 \frac{r_0}{r - r_0} \right) A_0^{(0)} \,, \tag{2.37}$$

which is smaller than $A_x^{(0)}$ only if the condition (2.33) is satisfied.

Thus, we have established that the assumptions underlying our derivation of Fick's law are valid, assuming that the location of the stretched horizon is chosen to satisfy (2.28) and (2.33).

3. Shear viscosity

So far we have found that small fluctuations of the stretched horizon have properties, which can be viewed as corresponding to diffusion of a conserved charge in simple fluids. We now turn to the next simplest hydrodynamic mode — the shear mode.

In principle, it should be possible to define the energy-momentum tensor $T^{\mu\nu}$ living on the stretched horizon in a manner similar to eq. (2.8) and show that $T^{\mu\nu}$ is conserved and, with a suitable choice for the location of the stretched horizon, satisfies the constitutive equations. We shall follow a simpler route. We will show that the corresponding fluctuations of the metric obey a diffusive dispersion law, $\omega = -i\mathcal{D}q^2$, and identify $\mathcal{D}$ with

$\eta/(\epsilon + P)$ in the dual theory. Here η is the shear viscosity, ϵ and P are the equilibrium energy density and pressure (see appendix B). We will call $\mathcal{D}$ the shear mode diffusion constant.

We write fluctuations of the $(p+2)$-dimensional background (2.2) as $g_{\mu\nu} + h_{\mu\nu}$ and consider only those perturbations which depend on t, r, $x = x^1$, with only two non-vanishing components of $h_{\mu\nu}$

$$h_{ty} = h_{ty}(t,x,r)\,, \qquad h_{xy} = h_{xy}(t,x,r)\,, \tag{3.1}$$

where $x \equiv x^1$, $y \equiv x^2$ (assuming the black brane is extended along at least two spatial dimensions). This is motivated by the fact that hydrodynamic shear mode describes decay of the fluctuations in transverse momentum density of the fluid. We shall call perturbations of the type (3.1) gravitational shear perturbations.

The linearized field equations for h_{ty} and h_{xy} decouple from all other modes and so they can be treated separately.[4] The easiest way to find the field equations for the gravitational shear perturbations is to notice that since h_{ty} and h_{xy} do not depend on y, they can be viewed as zero modes of the Kaluza-Klein reduction on a circle along the y direction. From the standard formulas of Kaluza-Klein compactification one finds that the fields $A_0 = (g_{xx})^{-1} h_{ty}$, $A_x = (g_{xx})^{-1} h_{xy}$ satisfy Maxwell's equations on a $(p+1)$-dimensional background, whose metric components are

$$\bar{g}_{\alpha\beta} = g_{\alpha\beta}\, (g_{xx})^{\frac{1}{p-1}}\,. \tag{3.2}$$

The indices α, β run from 0 to p. Indeed, the gravitational action contains the following piece

$$\sqrt{-g}\,\mathcal{R}_{(g)} \to -\frac{1}{4}\sqrt{-\bar{g}}\,F_{\alpha\beta}F_{\gamma\delta}\,\bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}\,(g_{xx})^{\frac{p}{p-1}} + \dots\,, \tag{3.3}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and the only non-zero components are F_{tx}, F_{tr}, and F_{xr}.

Thus the problem is reduced to that of abelian vector field considered in the previous section,[5] with the identification $1/g_{\rm eff}^2 = (g_{xx})^{\frac{p}{p-1}}$. The shear mode damping constant $\mathcal{D}$ is given by the result (2.27), evaluated with $\bar{g}_{\alpha\beta}$. Using the relation (3.2) between $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, one finds

$$\begin{aligned}\mathcal{D} &= \frac{\sqrt{-g(r_0)}}{\sqrt{-g_{00}(r_0)\,g_{rr}(r_0)}}\int_{r_0}^{\infty} dr\,\frac{-g_{00}(r)\,g_{rr}(r)}{g_{xx}(r)\sqrt{-g(r)}}\\ &= \frac{\sqrt{-G(r_0)}}{\sqrt{-G_{00}(r_0)\,G_{rr}(r_0)}}\int_{r_0}^{\infty} dr\,\frac{-G_{00}(r)\,G_{rr}(r)}{G_{xx}(r)\sqrt{-G(r)}}\,.\end{aligned} \tag{3.4}$$

In writing the last expression, the relation between components of the black brane metric $G_{\mu\nu}$, and its dimensionally reduced version $g_{\mu\nu}$ was used, $g_{\mu\nu} = G_{\mu\nu}Z^{d/p}$.

[4] By a choice of gauge one can set h_{ry} to zero.

[5] When solving for gravitational shear perturbations in 10 or 11-dimensional supergravity, one can consistently set fluctuations of all other fields to zero.

Using thermodynamic relation $\epsilon + P = Ts$, one finds shear viscosity

$$\eta = s\,T \frac{\sqrt{-G(r_0)}}{\sqrt{-G_{00}(r_0)\,G_{rr}(r_0)}} \int\limits_{r_0}^{\infty} dr\, \frac{-G_{00}(r)\,G_{rr}(r)}{G_{xx}(r)\sqrt{-G(r)}} \,. \tag{3.5}$$

This expression gives shear viscosity in terms of components of the black brane metric (2.1), its Hawking temperature T and its entropy s per unit p-dimensional volume.

The formula (3.5), and the corresponding one for the diffusion constant,

$$D = \frac{\sqrt{-G(r_0)}\,Z(r_0)}{G_{xx}(r_0)\sqrt{-G_{00}(r_0)\,G_{rr}(r_0)}} \int\limits_{r_0}^{\infty} dr\, \frac{-G_{00}(r)\,G_{rr}(r)}{\sqrt{-G(r)}\,Z(r)} \,. \tag{3.6}$$

are the main results of this paper.

4. Applications

In this section, we shall apply the general formulas (3.5) and (3.6) to different gravitational backgrounds. For near-extremal D3, M2 and M5 branes, the results coincide with those found previously in refs. [10, 11, 12] from the AdS/CFT correspondence.

4.1 Near-extremal D3-branes

The metric of a stack of N non-extremal D3 branes in type-IIB supergravity is given in the near-horizon region by

$$ds^2 = \frac{r^2}{R^2}\left(-f(r)\,dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{R^2}{r^2 f(r)}\,dr^2 + R^2 d\Omega_5^2 \,. \tag{4.1}$$

Here R is a constant, which depends on the number of D3 branes, $R \propto N^{1/4}$, and $f(r) = 1 - r_0^4/r^4$. (This metric is an important example, because for type-IIB supergravity on this background, the holographically dual theory is known explicitly.) The Hawking temperature of the background metric (4.1) is $T = r_0/\pi R^2$, and the entropy per unit (three-dimensional) volume is $s = \frac{\pi^2}{2} N^2 T^3$.

Applying eqs. (3.6), (3.5), one finds the corresponding diffusion constant,

$$D = \frac{1}{2\pi T} \,, \tag{4.2}$$

and the shear viscosity

$$\eta = \frac{\pi}{8} N^2 T^3 \,. \tag{4.3}$$

The holographic dual theory for type-IIB supergravity on the background (4.1) is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group SU(N), in the limit of large-N and large 't Hooft coupling [4]. This field theory lives in $3+1$ (infinite, flat) dimensions, and must be taken at finite temperature, equal to the Hawking temperature of the gravitational background (4.1). Long-time, long-distance behavior in this theory is governed by conventional hydrodynamics,[6] with hydrodynamic variables being just densities

[6] For systematic discussion of hydrodynamic fluctuations in supersymmetric theories, see [15].

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of conserved charges. This theory has SU(4) global symmetry current (R-current) which therefore must relax diffusively. The Fick's law for this current is $\boldsymbol{j}_a = -D_{ab}\boldsymbol{\nabla} j_b^0$, where a, b are adjoint SU(4) indices, which run from 1 to 15. When thermal equilibrium state respects the SU(4) symmetry (no chemical potentials for R-charges), the matrix of diffusion constants must be an invariant of the group: $D_{ab} = D_R\, \delta_{ab}$. Thus the R-charge diffusion is characterized by only one constant D_R, which can be computed by using the AdS/CFT approach [11]. The result of the AdS/CFT calculation of D_R is exactly equal to (4.2). Likewise, the result (4.3) exactly coincides with the shear viscosity of the supersymmetric Yang-Mills plasma, as computed by AdS/CFT methods [10, 11].

In this example, the ratio of shear viscosity to entropy density is $\eta/s = 1/(4\pi)$.

4.2 M2 branes

The metric of a stack of N non-extremal M2 branes in the eleven-dimensional supergravity is given by

$$ds^2 = H^{-2/3}(-f dt^2 + d\boldsymbol{x}^2) + H^{1/3}(f^{-1}dr^2 + r^2 d\Omega_7^2)\,, \tag{4.4}$$

where $H = 1 + R^6/r^6$, $f = 1 - r_0^6/r^6$, and R is a constant, which depends on the number of M2 branes, $R \propto N^{1/6}$. The Hawking temperature of the background metric (4.4) is

$$T = \frac{3}{2\pi r_0}\,\frac{1}{H^{1/2}(r_0)}\,. \tag{4.5}$$

Using eqs. (3.6) and (3.5), we find

$$D = \frac{r_0}{8}\, H^{3/2}(r_0)\; {}_2F_1\left(1, \frac{4}{3}; \frac{7}{3}; -\frac{R^6}{r_0^6}\right), \tag{4.6a}$$

$$\frac{\eta}{\epsilon + P} = \frac{r_0}{6}\, H^{1/2}(r_0) = \frac{1}{4\pi T}\,, \tag{4.6b}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function. In the near-extremal limit $r_0 \ll R$, the expression (4.6a) gives[7] $D = 3/(4\pi T)$.

The details of the CFT holographically dual to the near-horizon limit of the background (4.4) are not known explicitly. However, it is known that its symmetry algebra includes (among other things) translation symmetry, and a global SO(8) R-symmetry. Thus it makes sense to consider the long-time relaxation of charge densities of the corresponding conserved currents in the dual theory. The Minkowski AdS/CFT recipe [16, 17] can be used to calculate correlation functions of the energy-momentum tensor and R-currents in this theory at finite temperature, in the low-frequency limit. The AdS/CFT calculation [12] shows that the R-current relaxes diffusively, with the diffusion constant equal to $3/(4\pi T)$, and that the shear mode damping constant is $1/4\pi T$. Thus we see again that the general formulas (3.5) and (3.6) reproduce previous AdS/CFT results.

In this example, we also have $\eta/s = (4\pi)^{-1}$, independent of the ratio r_0/R.

[7] One can either use the full metric (4.4) in the formula (3.6) and then take the near-extremal limit $r_0 \ll R$, or directly substitute the near-horizon limit of the metric (4.4) in eq. (3.6), with the same result.

4.3 M5 branes

The metric of a stack of N non-extremal M5 branes in the eleven-dimensional supergravity is given by

$$ds^2 = H^{-1/3}(-f dt^2 + d\boldsymbol{x}^2) + H^{2/3}(f^{-1}dr^2 + r^2 d\Omega_4^2)\,, \tag{4.7}$$

where $H = 1 + R^3/r^3$, $f = 1 - r_0^3/r^3$, and R is a constant, which depends on the number of M5 branes, $R \propto N^{1/3}$. The Hawking temperature of the background metric (4.7) is

$$T = \frac{3}{4\pi r_0}\frac{1}{H^{1/2}(r_0)}\,. \tag{4.8}$$

Applying eqs. (3.6) and (3.5), we get

$$D = \frac{r_0}{5}\, H^{3/2}(r_0)\; {}_2F_1\left(1, \frac{5}{3}; \frac{8}{3}; -\frac{R^3}{r_0^3}\right), \tag{4.9a}$$

$$\frac{\eta}{\epsilon + P} = \frac{r_0}{3}\, H^{1/2}(r_0) = \frac{1}{4\pi T}\,. \tag{4.9b}$$

In the near-extremal limit $r_0 \ll R$, the expression (4.9a) gives $D = 3/(8\pi T)$. Analogously to the previous example of M2 branes, these values reproduce the AdS/CFT results of ref. [12]. Again, in this example $\eta/s = (4\pi)^{-1}$.

4.4 Dp branes

Black p-brane metrics are solutions to the low-energy string theory equations of motion [18]. The metric ($p < 7$) in the Einstein frame reads

$$ds_E^2 = H^{-\frac{7-p}{8}}\left(-f dt^2 + dx_1^2 + \cdots + dx_p^2\right) + H^{\frac{p+1}{8}}\left(f^{-1}dr^2 + r^2 d\,\Omega_{8-p}^2\right), \tag{4.10}$$

where

$$H = 1 + \frac{R^{7-p}}{r^{7-p}}\,, \qquad f = 1 - \frac{r_0^{7-p}}{r^{7-p}}\,. \tag{4.11}$$

The Ramond-Ramond field strength is given by

$$F_{r012\ldots p} = (p-7)\frac{R^{(7-p)/2}\sqrt{r_0^{7-p} + R^{7-p}}}{H^2(r)\, r^{8-p}}\,, \tag{4.12}$$

and the dilaton is

$$e^{\Phi} = H^{(3-p)/4}(r)\,. \tag{4.13}$$

The Hawking temperature is

$$T = \frac{7-p}{4\pi r_0} H^{-1/2}(r_0)\,. \tag{4.14}$$

According to the AdS/CFT dictionary, the isometries of the $(8-p)$-dimensional sphere in the metric correspond to a SO$(9-p)$ global R-symmetry of the dual theory. Fluctuations of the metric with one index along the brane and another one along the sphere become, upon dimensional reduction on S^{8-p}, gauge fields propagating in the $(p+2)$-dimensional bulk which couple to the R-symmetry currents on the $(p+1)$-dimensional flat boundary.

Dimensionally reducing the full Dp-brane metric (4.10) we obtain the following metric

$$ds^2_{p+2} = H^{\frac{1}{p}} r^{\frac{2(8-p)}{p}} \left(-f dt^2 + dx_1^2 + \cdots + dx_p^2\right) + H^{\frac{1}{p}+1} r^{\frac{2(8-p)}{p}} f^{-1} dr^2 \,. \tag{4.15}$$

The action of the gauge field is

$$S \sim \int d^{p+2}x \, \sqrt{-g_{p+2}} \frac{1}{g^2_{\text{eff}}} F^2_{\mu\nu} \,, \tag{4.16}$$

where constant normalization factors are ignored. The effective gauge coupling constant depends on the radial coordinate,

$$g^2_{\text{eff}}(r) = H^{-\frac{p+1}{p}} r^{-\frac{16}{p}} \,. \tag{4.17}$$

Applying the formulas (2.27) and (3.5) we find

$$D = \frac{r_0^{8-p} H^{3/2}(r_0)}{2R^{7-p}} \, {}_2F_1 \left(1, -\frac{2}{7-p}; \frac{5-p}{7-p}; -\frac{r_0^{7-p}}{R^{7-p}}\right), \tag{4.18a}$$

$$\frac{\eta}{\epsilon + P} = \frac{r_0}{7-p} H^{1/2}(r_0) = \frac{1}{4\pi T} \,. \tag{4.18b}$$

We find again that $\eta/s = (4\pi)^{-1}$, and in the near-extremal regime $r_0 \ll R$ the R-charge diffusion constant is $D = (7-p)/(8\pi T)$. We shall reproduce these results in section 5 by using the AdS/CFT prescription.

4.5 A universal lower bound on η/s?

In all examples considered so far, there is a remarkable regularity: the ratio of shear viscosity to entropy density is always equal to $(4\pi)^{-1}$. This ratio is independent of the parameters of the metric and even of the dimensionality of space-time. This fact becomes more interesting if we notice that for any spacetime dimension the ratio η/s has the dimension of the Planck constant. In the SI system of units the ratio η/s in all cases considered is

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B} \approx 6.08 \times 10^{-13} \text{ K} \cdot \text{s} \,. \tag{4.19}$$

One may suspect that the constancy of η/s is an inherent property of classical gravity, and hence eq. (4.19) should be valid for any theory with gravitational dual description. This includes $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the regime of large N, large 't Hooft coupling, and its deformations.

We have checked this conclusion on the example of the supergravity solution recently constructed by Buchel and Liu [19] . It describes the finite-temperature $\mathcal{N} = 2^*$ SU(N) gauge theory at large 't Hooft coupling. The solution is a non-extremal generalization of the Pilch-Warner RG flow [20] describing the mass deformation of the $\mathcal{N} = 4$ SYM by a hypermultiplet term. In addition to the temperature T, new parameters, the hypermultiplet masses m_b, m_f, appear in the solution. The leading corrections to the metric are known in the high-temperature regime $m_b/T \ll 1$, $m_f/T \ll 1$, with $m_b/T = 0$, $m_f/T = 0$ corresponding to the near-extremal black three-brane solution. In appendix C we show that the leading corrections to η/s vanish identically.

One may suspect that the number $\hbar/(4\pi k_B)$ is somehow special. We conjecture that it is a lower bound on η/s. Since this lower bound does not contain the speed of light c, we suggest that this is a lower bound for all systems, *including non-relativistic ones.* This means we can check the conjecture on common substances where both viscosity and entropy density have been measured.

Take, for example, water under normal conditions (298.15 K, atmospheric pressure). From the table [21] one finds $s \approx 3.9 \times 10^6$ J $\cdot$ K$^{-1} \cdot$ m^{-3} and $\eta \approx 0.89$ mPa $\cdot$ s. This means $\eta/s \approx 2.3 \times 10^{-10}$ K $\cdot$ s, which is by a factor of 400 larger than (4.19). We have checked other common substances and found that the ratio η/s is always larger than $\hbar/(4\pi k_B)$. The minimum value we found is for liquid ^{4}He at 1 MPa and 10 K, for which $\eta \approx 3.07$ µPa $\cdot$ s and $s \approx 4.95 \times 10^5$ J $\cdot$ K$^{-1} \cdot$ m^{-3} [21], and the ratio η/s is still larger than (4.19) by a factor of 10.

Let us now argue that in weakly coupled theories the ratio η/s should be much larger than 1 and thus satisfies the lower bound. Weakly coupled systems can be described as dilute gases of weakly interacting quasiparticles. Finite-temperature $\lambda\phi^4$ theory with $\lambda \ll 1$ and gauge theories with small gauge coupling $g \ll 1$ belong to this class. Nonrelativistic Bose and Fermi liquids at temperatures much below quantum degeneracy are also weakly coupled since they can be described as dilute gases of quasiparticles (phonons in the case of Bose liquids, and dressed fermionic quasiparticles in the case of Fermi liquids).

The entropy density s of a weakly coupled system is proportional to the number density of quasiparticles n,

$$s \sim k_B n \,. \tag{4.20}$$

The shear viscosity is proportional to the product of the energy density and the mean free time (time between collisions) τ

$$\eta \sim n\epsilon\tau \,, \tag{4.21}$$

where ϵ is the average energy per particle (which is of the order of the temperature T). Therefore

$$\frac{\eta}{s} \sim \frac{\epsilon\tau}{k_B} \,. \tag{4.22}$$

Now, in order for the quasiparticle picture to be valid, the width of the quasiparticles must be small compared to their energies, i.e., one should have

$$\frac{\hbar}{\tau} \ll \epsilon \tag{4.23}$$

which means that

$$\frac{\eta}{s} \gg \frac{\hbar}{k_B} \,. \tag{4.24}$$

Thus we see that (in units $\hbar = k_B = 1$) the ratio η/s is large at weak coupling. Theories whose duals are described by supergravity are typically strongly coupled, thus naturally having η/s of order 1. It is still puzzling that this ratio takes the same value of $(4\pi)^{-1}$ for all such theories.

5. Transport coefficients from AdS/CFT

We have seen that the two simple formulas (3.5) and (3.6) reproduce the known results for the transport coefficients of theories living on D3, M2 and M5 branes at finite temperature. For the general case of Dp brane there is no previous calculation to compare our result with. Here we show how to find the R-charge diffusion rate and the viscosity by directly calculating the correlation functions of the R-current and the stress-energy tensor following the prescription of ref. [16]. (At zero temperature, such correlators were considered in ref. [22].)

The metric of black Dp-branes is given by eq. (4.10). In this section we work exclusively in the near-horizon limit $r \ll R$. It will be convenient to introduce a new radial variable ($u = r_0/r$ for metrics with even p and $u = r_0^2/r^2$ for the ones with odd p), in terms of which the near-horizon metric for even and odd p respectively becomes

$$ds_E^2 = \left(\frac{r_0}{uR}\right)^{\frac{(7-p)^2}{8}} \left(-f dt^2 + dx_1^2 + \cdots + dx_p^2\right) + \left(\frac{uR}{r_0}\right)^{\frac{(7-p)(p+1)}{8}} \frac{r_0^2}{u^2} \left(\frac{du^2}{u^2 f} + d\Omega_{8-p}^2\right), \tag{5.1}$$

where $f(u) = 1 - u^{7-p}$ (even p) and

$$ds_E^2 = \left(\frac{r_0^2}{uR^2}\right)^{\frac{(7-p)^2}{16}} \left(-f dt^2 + dx_1^2 + \cdots + dx_p^2\right) + \left(\frac{uR^2}{r_0^2}\right)^{\frac{(7-p)(p+1)}{16}} \frac{r_0^2}{u} \left(\frac{du^2}{4u^2 f} + d\Omega_{8-p}^2\right), \tag{5.2}$$

where $f(u) = 1 - u^{\frac{7-p}{2}}$ (odd p).

The dimensionally reduced metric (4.15), correspondingly, takes the form

$$ds_E^2 = R^{\frac{7-p}{p}} \left(\frac{r_0}{u}\right)^{\frac{9-p}{p}} \left(-f dt^2 + dx_1^2 + \cdots + dx_p^2\right) + R^{6-p+7/p} \left(\frac{r_0}{u}\right)^{p+\frac{9}{p}-6} \frac{du^2}{u^2 f} \tag{5.3}$$

for even p and

$$ds_E^2 = R^{\frac{7-p}{p}} \left(\frac{r_0^2}{u}\right)^{\frac{9-p}{2p}} \left(-f dt^2 + dx_1^2 + \cdots + dx_p^2\right) + R^{6-p+7/p} \left(\frac{r_0^2}{u}\right)^{\frac{p}{2}+\frac{9}{2p}-3} \frac{du^2}{4u^2 f} \tag{5.4}$$

for odd p, with the same functions $f(u)$ as in eqs. (5.1), (5.2).

For the backgrounds (5.1), (5.2) we now compute the R-current retarded correlators in dual gauge theories, as well as the correlators of the shear-mode components of the stress-energy tensor. For frequencies and momenta much smaller than the temperature we find that each of these correlators has a diffusion-type pole with a specific value of the diffusion constant. The calculations are very similar to those done for the D3-brane background in ref. [11] and for M-branes in ref. [12], where details of the approach can be found. Our results are summarized in table 1.

5.1 R-charge diffusion

In computing two-point functions of R-currents in the large-N limit, it is sufficient to treat the bulk gauge fields as abelian ones, ignoring the self-interactions [24]. Accordingly, to find the R-charge diffusion mode, we consider Maxwell's equations

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}\right) = 0 \tag{5.5}$$

Brane	s	$\frac{\eta}{\epsilon+P}$	D	Brane	s	$\frac{\eta}{\epsilon+P}$	D
D1	$\sim \frac{N^2T^2}{\sqrt{g_{YM}^2 N}}$	—	$\frac{3}{4\pi T}$	D5	$s = \frac{\epsilon}{T}$	$\frac{1}{4\pi T}$	$\frac{1}{4\pi T}$
D2	$\sim \frac{N^2T^{7/3}}{(g_{YM}^2 N)^{1/3}}$	$\frac{1}{4\pi T}$	$\frac{5}{8\pi T}$	D6	$\sim \frac{N^2}{(g_{YM}^2 N)^3 T^3}$	$\frac{1}{4\pi T}$	$\frac{1}{8\pi T}$
D3	$\sim N^2T^3$	$\frac{1}{4\pi T}$	$\frac{1}{2\pi T}$	M2	$\sim N^{3/2}T^2$	$\frac{1}{4\pi T}$	$\frac{3}{4\pi T}$
D4	$\sim g_{YM}^2 N^3 T^5$	$\frac{1}{4\pi T}$	$\frac{3}{8\pi T}$	M5	$\sim N^3T^5$	$\frac{1}{4\pi T}$	$\frac{3}{8\pi T}$

Table 1: Entropy density [23] and diffusion constants for black p- and M-branes. There is no gravitational shear mode for D1 brane. For D5 brane, the entropy is linear in energy.

on the backgrounds (5.3), (5.4). Here to simplify notations we denote $\sqrt{-g} \equiv \sqrt{-g_{p+2}}/g_{\text{eff}}^2$. We choose the gauge $A_u = 0$ and restrict ourselves to fields which depend only on the u, t and $x \equiv x^1$,

$$A_\mu(u,t,z) = \int \frac{d\omega\, dq}{(2\pi)^2}\, e^{-i\omega t + iqx} A_\mu(\omega, q, u)\,. \tag{5.6}$$

Then for the components A_t and A_x one finds the system of equations

$$g^{tt}\omega A_t' - q g^{xx} A_x' = 0\,, \tag{5.7a}$$
$$\partial_u\left(\sqrt{-g} g^{tt} g^{uu} A_t'\right) - \sqrt{-g} g^{tt} g^{xx}\left(\omega q A_x + q^2 A_t\right) = 0\,, \tag{5.7b}$$
$$\partial_u\left(\sqrt{-g} g^{xx} g^{uu} A_t'\right) - \sqrt{-g} g^{tt} g^{xx}\left(\omega q A_t + \omega^2 A_x\right) = 0\,, \tag{5.7c}$$

where prime denotes the derivative with respect to u. The equations for other components of A_μ decouple, and thus these components can be consistently set to zero. Eqs. (5.7a) and (5.7b) can be combined to give a closed-form equation for A_t'

$$\frac{d}{du}\left[\frac{\partial_u(\sqrt{-g} g^{tt} g^{uu} A_t')}{\sqrt{-g} g^{tt} g^{xx}}\right] - \left(\frac{g^{tt}}{g^{xx}}\omega^2 + q^2\right) A_t' = 0\,. \tag{5.8}$$

According to the method developed in ref. [11], one solves eq. (5.8) for A_t' with the incoming-wave boundary condition at $u = 1$. This can be done in the hydrodynamic regime $\omega/T \ll 1$, $q/T \ll 1$. The obtained A_t' depends on a single overall normalization constant which can be fixed from eq. (5.7b) and the boundary values $A_t^0 = \lim_{u\to 0} A_t(u)$, $A_z^0 = \lim_{u \to 0} A_z(u)$. One then finds A_x' from eq. (5.7a). This is sufficient for computing the on-shell classical action, which is proportional to $A_t A_t' - A_x A_x'$ at $u = 0$. Taking derivatives of the action with respect to A_t^0 and A_x^0 one then obtains the current correlators.

In backgrounds (5.3), (5.4), eq. (5.8) becomes

$$\frac{d}{du}\left[(1-u^\nu)\, u^\alpha \frac{d}{du}\left(u^\beta A_t'\right)\right] + \left(\frac{\mathfrak{w}^2}{1-u^\nu} - \mathfrak{q}^2\right) A_t' = 0\,, \tag{5.9}$$

where for p even

$$\nu_{\text{even}} = 7 - p\,, \qquad \alpha_{\text{even}} = p-2\,, \qquad \beta_{\text{even}} = -1\,, \tag{5.10}$$

while for p odd

$$\nu_{\text{odd}} = \frac{7-p}{2}\,, \qquad \alpha_{\text{odd}} = \frac{p-1}{2}\,, \qquad \beta_{\text{odd}} = 0\,. \tag{5.11}$$

The dimensionless parameters $\mathfrak{w}$, $\mathfrak{q}$ are defined as

$$\mathfrak{w} \equiv \frac{\nu\,\omega}{4\,\pi\,T}\,, \qquad \mathfrak{q} \equiv \frac{\nu\,q}{4\,\pi\,T}\,. \tag{5.12}$$

Near the horizon, two local independent solutions of eq. (5.9) are $A_t' \sim (u-1)^{a_\pm}$, where $a_\pm = \pm i\mathfrak{w}/\nu$. The "incoming wave" boundary condition corresponds to choosing a_- as the correct exponent. Solving eq. (5.9) perturbatively in $\mathfrak{w}$, $\mathfrak{q}$ we find

$$A_t' = C(u-1)^{-\frac{i\mathfrak{w}}{\nu}} u^{-\beta} \left(1 + \mathfrak{w}\, F^{(p)}(u) + \mathfrak{q}^2\, G^{(p)}(u) + \cdots \right), \tag{5.13}$$

here $F^{(p)}(u)$ and $G^{(p)}(u)$ are explicitly known (but rather cumbersome) functions, independent of $\mathfrak{w}$, $\mathfrak{q}$. The constant C can be found from eq. (5.7b) and the boundary values of fields at $u=0$. We have

$$Ce^{\frac{\pi\mathfrak{w}}{\nu}} \left(i\,\mathfrak{w} + \frac{7-p}{2\nu}\,\mathfrak{q}^2 + \cdots \right) = \left(\mathfrak{w}\,\mathfrak{q}\,A_z^0 + \mathfrak{q}^2\,A_t^0\right), \tag{5.14}$$

where the ellipses represent terms of higher order in $\mathfrak{w}$, $\mathfrak{q}$. Here we first see the emergence of the diffusion pole: $C \sim (i\mathfrak{w} + \frac{7-p}{2\nu}\mathfrak{q}^2)^{-1}$. This pole will appear in the correlators. It corresponds to the diffusion constant

$$D = \frac{7-p}{8\pi T}\,. \tag{5.15}$$

5.2 Shear viscosity

The low-energy string theory equations of motion which give rise to the black p-brane backgrounds (4.10) can be deduced from the following Einstein-frame effective action

$$S = \int d^{10}x\sqrt{-g}\left(\mathcal{R} - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2n!}e^{a\Phi}F_n^2\right). \tag{5.16}$$

Here Φ is the dilaton, F_n is the Ramond-Ramond field strength form (or its dual), and values of n and a are related to p for a given p-brane background. The equations of motion that follow from eq. (5.16) are

$$\mathcal{R}_{\mu\nu} = \frac{1}{2}\,\partial_\mu\Phi\,\partial_\nu\Phi + \frac{1}{2(n-1)!}\,e^{a\Phi}\left(F_{\mu\ldots}F_\nu^{\ldots} - \frac{n-1}{8n}F^2\,g_{\mu\nu}\right), \tag{5.17a}$$

$$\nabla_\mu\left(e^{a\Phi}\,F^{\mu\ldots}\right) = 0\,, \tag{5.17b}$$

$$\Box\Phi = \frac{a}{2n!}\,e^{a\Phi}F^2\,. \tag{5.17c}$$

To find the viscosity, we can substitute the perturbed metric $g_{\mu\nu} + h_{\mu\nu}$ into eq. (5.17a), solve the resulting equations, and then use the Minkowski AdS/CFT correspondence to find the retarded correlators of the appropriate components of the stress-energy tensor in the dual gauge theory. The case of $p=3$ was treated in this way in ref. [11].

Alternatively, we may use the fact that, as noted in section 3, the shear gravitational perturbations satisfy Maxwell equations with a coordinate-dependent effective coupling in a dimensionally reduced background. Thus the problem is reduced to the one treated in section 5.1: the system of the effective Maxwell's equations is given by (5.7a)–(5.7c), with the metrics (5.1), (5.2) for even and odd p respectively, $\sqrt{-g}$ replaced by $\sqrt{-g}/g_{\rm eff}^2 = g_{xx}\sqrt{-g}$ and $A_t = (g_{xx})^{-1}h_{ty}$, $A_x = (g_{xx})^{-1}h_{xy}$. Then the equation for the effective A_t' still has the form (5.9), but with a new set of parameters,

$$\nu_{\rm even} = 7 - p\,, \qquad \alpha_{\rm even} = 3\,, \qquad \beta_{\rm even} = p - 6\,, \tag{5.18}$$

$$\nu_{\rm odd} = \frac{7-p}{2}\,, \qquad \alpha_{\rm odd} = 2\,, \qquad \beta_{\rm odd} = \frac{p-5}{2}\,. \tag{5.19}$$

The subsequent calculations follow the Maxwell case verbatim, and in the end we find the hydrodynamic pole in shear part of the correlator of $T^{\mu\nu}$ at $\mathfrak{w} = -i\mathfrak{q}^2/\nu + \cdots$. The dispersion relation is $\omega = -i\mathcal{D}q^2 + \cdots$, where $\mathcal{D} = 1/4\pi T$ is independent of p, hence $\eta/s = 1/4\pi$.

6. Discussion

Let us briefly summarize. The analysis of small fluctuations around a black-brane background shows that in the low frequency, long-wavelength limit, there exist perturbations that correspond to diffusive hydrodynamic processes on the stretched horizon. This result indicates that the degrees of freedom holographically dual to a gravitational theory on a black-brane background (whatever their microscopic nature is) must have a rather conventional hydrodynamic limit at large distances and long times.

We have also derived general expressions for the diffusion constant and shear viscosity in terms of the components of the background metric. For near-extremal Dp and M-branes, these formulas reproduce the results of the direct AdS/CFT calculations.[8] For all non-extremal supergravity backgrounds considered in the paper we found that the ratio of shear viscosity to entropy density is a universal number[9] equal to $(4\pi)^{-1}$.

We find these results rather intriguing, and some future work is yet to be done. First, at the moment we lack crisp understanding of why our analysis of small perturbations in the vicinity of the horizon yields the same result as the AdS/CFT calculation.

Another question to be addressed is the complete treatment of gravitational perturbations. Indeed, the fluctuations of the stretched horizon must encode the full set of hydrodynamic modes, at least in the known AdS/CFT examples. This means that methods similar to those used in section 2 should reproduce propagating modes (sound), in

[8] It should be stressed that the notion of diffusion (more generally, hydrodynamics) on stretched horizons is not limited to the particular string-theoretic realization of holography. It is encoded in the low-frequency part of the spectrum of quasinormal modes for a given gravitational background (see [25] for a discussion of quasinormal modes in this context).

[9] This observation traces back to the observation in [12] that shear viscosities and entropies of M-branes have the same N dependence. Curiously, the viscosity to entropy ratio is also equal to $1/4\pi$ in the pre-holographic "membrane paradigm" hydrodynamics [8, 9]: there, for a four-dimensional Schwarzschild black hole one has $\eta_{\rm m.p.} = 1/16\pi G_N$, while the Bekenstein-Hawking entropy is $s = 1/4G_N$.

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addition to the non-propagating shear and diffusive modes.[10] In particular, it is reasonable to expect that formulas analogous to (3.5) exist also for bulk viscosity and for the speed of sound.

It would be interesting to extend and generalize our calculations to include other nontrivial supergravity backgrounds, such as rotating branes [26], non-extremal Klebanov-Tseytlin solution [27], or the full (as opposed to the high-temperature limit) Buchel-Liu metric [19].

Another interesting question is the gravitational description of non-liner terms in hydrodynamic equations. Namely, the hydrodynamic constitutive relations for conserved currents contain in addition to the linear term (Fick's law), a non-linear convective term: $j^i = -D\,\partial^i j^0 + (\epsilon + P)^{-1}\, j^0\, \pi^i$. Here π^i is momentum density in the i-th direction; the last term is just charge density times velocity of the fluid. This term, which is quadratic in small fluctuations, is a direct consequence of Lorentz (or galilean) invariance of the microscopic theory.[11] The gravitational manifestation of the convective term may not be straightforward: it was shown in [15] that such non-linear terms give rise to $O(1/N^2)$ effects in correlation functions of conserved currents. In the context of AdS/CFT, this implies that gravity loop effects must be included.

Finally, we hope that analyzing the hydrodynamic regime of black branes will lead to a better conceptual understanding of holography.

Acknowledgments

P.K.K. thanks A. Karch and L. Yaffe for discussions and comments. A.O.S. would like to thank D. Birmingham, O. DeWolfe, G. Horowitz, L. Lindblom, A. Mikhailov, J. Polchinski, R. Roiban, A. Volovich, J. Walcher, and especially A. Buchel and C. Herzog for conversations, questions and comments, and the KITP at UC Santa-Barbara for its warm hospitality. P.K.K. is supported, in part, by DOE Grant No. DE-FG03-96ER40956. D.T.S. and A.O.S. are supported, in part, by DOE Grant No. DOE-ER-41132. The work of D.T.S. is supported, in part, by the Alfred P. Sloan Foundation.

A. Dimensional reduction

In this appendix we write down the formulas of the dimensional reduction scheme relevant for our discussion in the main text. More details can be found in [28]–[31].

The dimensional reduction of a D-dimensional pure Einstein theory is performed by using the ansatz

$$ds^2 = e^{2a\phi} ds_X^2 + e^{2b\phi} ds_Y^2\,, \tag{A.1}$$

where X is the lower-dimensional space and Y is the internal compactifying space with dimensions d_X and d_Y respectively ($d_X + d_Y = D$). In the reduced action, the lower-dimensional Einstein-Hilbert term and the kinetic term for the scalar ϕ will both have

[10] Sound waves from the AdS/CFT perspective were considered in [13, 14].

[11] Analogous non-linear terms also exist in the constitutive relations for stress T^{ij}.

canonical normalization, if we take

$$a = \sqrt{\frac{d_Y}{2(d_X - 2)(d_X + d_Y - 2)}}\,, \qquad b = -\sqrt{\frac{d_X - 2}{2d_Y(d_X + d_Y - 2)}}\,. \tag{A.2}$$

Considering vector-like fluctuations of the metric (A.1) (gravitons with one index along the X space and one index along the Y one), one observes that the normalization of the corresponding gauge fields is *not* canonical. Schematically,

$$\int \sqrt{-g}\,\mathcal{R} \to \int \sqrt{-g_X}\, e^{2(b-a)\phi}\, F_{\mu\nu}^2 + \cdots\,, \tag{A.3}$$

where constant prefactors are ignored. Thus in general the effective gauge coupling $1/g_{\text{eff}}^2 \sim e^{2(b-a)\phi}$ will be position-dependent.

B. Hydrodynamics

To make our presentation self-contained, here we briefly review the basic properties of hydrodynamic fluctuations. Hydrodynamics is an effective theory, which describes the dynamics of a thermal system on length and time scales which are large compared to any relevant microscopic scale. The degrees of freedom entering this theory, in the simplest cases, are the densities of conserved charges.

As an example, consider a translationally invariant theory in flat space, which has a conserved current j^μ, which corresponds to some global symmetry. The hydrodynamic degrees of freedom are the charge densities j^0, $\varepsilon \equiv T^{00}$, and $\pi^i \equiv T^{i0}$, where $T^{\mu\nu}$ is the energy-momentum tensor. The currents satisfy the conservation laws:

$$\partial_t j^0 = -\partial_i j^i\,, \tag{B.1}$$

$$\partial_t \varepsilon = -\partial_i \pi^i\,, \tag{B.2}$$

$$\partial_t \pi^i = -\partial_j T^{ij}\,. \tag{B.3}$$

The constitutive relations express the spatial currents j^i, T^{ij} in terms of j^0, ε, and π^i. To linear order

$$j_i = -D\,\partial_i j^0\,, \tag{B.4}$$

$$T_{ij} = \delta_{ij}\left(P + v_s^2 \delta\varepsilon\right) - \frac{\zeta}{\epsilon + P}\delta_{ij}\,\partial_k \pi_k - \frac{\eta}{\epsilon + P}\left(\partial_i \pi_j + \partial_j \pi_i - \frac{2}{p}\delta_{ij}\,\partial_k \pi_k\right). \tag{B.5}$$

Here $\epsilon = \langle\varepsilon\rangle$, $P = \frac{1}{p}\langle T_{ii}\rangle$ are equilibrium energy density and pressure, p is the number of spatial dimensions, $v_s = (\partial P/\partial\epsilon)^{1/2}$ is the speed of sound, and $\delta\varepsilon = (\varepsilon - \epsilon)$ is the fluctuation in energy density. The unknown kinetic coefficients D, ζ and η are diffusion constant, bulk and shear viscosities, respectively. The constitutive relation (B.4) is often called Fick's law.

The linearized constitutive relations together with conservation laws give the dispersion relations for hydrodynamic modes. Taking all charge densities to be proportional to $e^{-i\omega t + i\boldsymbol{q}\cdot\boldsymbol{x}}$, one finds: i) diffusive mode, $\omega = -i\boldsymbol{q}^2 D$, ii) shear mode, $\omega = -i\boldsymbol{q}^2\eta/(\epsilon + P)$, and iii) sound mode, $\omega = v_s q - i\boldsymbol{q}^2(\zeta + \frac{2p-2}{p}\eta)/(\epsilon + P)$.

Finally, linear response theory allows one to extract kinetic coefficients from singularities in equilibrium correlation functions of the corresponding conserved currents. This was used to find both diffusion constant and shear viscosity in some strongly-coupled theories from AdS/CFT recipe for correlation functions [11, 12]. We use this method in section 5.

C. Shear mode damping constant for the Buchel-Liu metric

Non-extremal deformation of the Pilch-Warner RG flow recently found by Buchel and Liu [19] is a gravity dual to the finite-temperature $\mathcal{N} = 2^*$ SU(N) gauge theory at large-N and large 't Hooft coupling. The five-dimensional metric is of the form

$$ds^2 = e^{2A}\left(-e^{2B}dt^2 + d\boldsymbol{x}^2\right) + dr^2\,, \tag{C.1}$$

where r is the radial coordinate in five dimensions. Functions $A(r)$, $B(r)$ satisfy supergravity equations of motion governing the flow. The system of equations (which also involves two r-dependent scalars) is given explicitly by eqs. (3.17) of [19]. Following [19] we introduce a new radial variable, $y = e^B$, $y \in [0,1]$, with $y = 0$ corresponding to the position of the horizon. The metric becomes

$$ds^2 = e^{2A}\left(-y^2dt^2 + d\boldsymbol{x}^2\right) + dr^2\left(\frac{\partial r}{\partial y}\right)^2 . \tag{C.2}$$

Using eqs. (3.17) of [19] one finds the following expression involving the jacobian $\partial y/\partial r$

$$\left(\frac{\partial y}{\partial r}\right)^2\left((A')^2 + \frac{1}{2y}A' - \left(\frac{\rho'}{\rho}\right)^2 - \frac{1}{3}(\chi')^2\right) = -\frac{1}{3}\mathcal{P}\,, \tag{C.3}$$

where prime denotes the derivative with respect to y. Functions $\rho(y)$ and $\chi(y)$ are the two scalars satisfying supergravity equations of motion, and $\mathcal{P}$ is their potential,

$$\mathcal{P} = \frac{1}{12}\left(\frac{1}{\rho^2} - \rho^4\cosh 2\chi\right)^2 + \frac{1}{16}\rho^8\sinh^2 2\chi - \frac{1}{3}\left(\frac{1}{\rho^2} + \frac{\rho^4}{2}\cosh 2\chi\right)^2 . \tag{C.4}$$

Applied to metric (C.2), our formula (3.4) for the shear mode damping constant reads

$$\mathcal{D} = e^{3A(0)}\int_0^1 \frac{ye^{-4A(y)}}{|\partial y/\partial r|}\,. \tag{C.5}$$

Knowing $A(y)$, $\rho(y)$ and $\chi(y)$, one can in principle determine $\mathcal{D}$ using eqs. (C.4) and (C.5). Unfortunately, the system of second order differential equations involving those functions is too complicated. Buchel and Liu [19] were able to solve it perturbatively using $\alpha_1 \propto (m_b/T)^2 \ll 1$ and $\alpha_2 \propto m_f/T \ll 1$, where m_b and m_f are masses of the bosonic and fermionic components of the $\mathcal{N} = 2$ hypermultiplet, as small parameters. To the leading order in α_1, α_2 the solution is [19]

$$\begin{aligned} A(y) &= \hat{\alpha} - \frac{1}{4}\ln(1-y^2) + \alpha_1^2 A_1(y) + \alpha_2^2 A_2(y)\,,\\ \rho(y) &= 1 + \alpha_1\rho_1(y)\,,\\ \chi(y) &= \alpha_2\chi_2(y)\,, \end{aligned} \tag{C.6}$$

where the scalars $\rho_1(y)$ and $\chi_2(y)$ obey linear differential equations

$$(1-y^2)^2 \left(y\,\rho_1'\right)' + y\,\rho_1 = 0\,, \tag{C.7}$$
$$(1-y^2)^2 \left(y\,\chi_2'\right)' + \frac{3}{4}\,y\,\chi_2 = 0\,, \tag{C.8}$$

and functions $A_1(y)$, $A_2(y)$ can be found by solving

$$y(1-y^2)\,A_1'' - (1+3y^2)\,A_1' + 4y(1-y^2)\,\left(\rho_1'\right)^2 = 0\,, \tag{C.9}$$
$$y(1-y^2)\,A_2'' - (1+3y^2)\,A_2' + \frac{4}{3}y(1-y^2)\,\left(\chi_2'\right)^2 = 0\,. \tag{C.10}$$

The scalars $\rho_1(y)$ and $\chi_2(y)$ are expressed in terms of the hypergeometric function

$$\rho_1 = (1-y^2)^{1/2}\;{}_2F_1(\tfrac{1}{2},\tfrac{1}{2},1;y^2)\,, \tag{C.11}$$
$$\chi_2 = (1-y^2)^{3/4}\;{}_2F_1(\tfrac{3}{4},\tfrac{3}{4},1;y^2)\,, \tag{C.12}$$

and functions A_1, A_2 are given by

$$\begin{aligned} A_1 &= \xi_1 - 4\int_0^y \frac{z\,dz}{(1-z^2)^2}\left(\frac{8-\pi^2}{2\pi^2} + \int_0^z dx\,\left(\frac{\partial\rho_1}{\partial x}\right)^2\frac{(1-x^2)^2}{x}\right), \\ A_2 &= \xi_2 - \frac{4}{3}\int_0^y \frac{z\,dz}{(1-z^2)^2}\left(\frac{8-3\pi}{8\pi} + \int_0^z dx\,\left(\frac{\partial\chi_2}{\partial x}\right)^2\frac{(1-x^2)^2}{x}\right). \end{aligned} \tag{C.13}$$

To the leading order in α_1, α_2, integration constants $\hat{\alpha}$, ξ_1, ξ_2 are related to the Hawking temperature of the black brane background by

$$T = \frac{1}{2\pi}e^{\hat{\alpha}}\left[1 + \alpha_1^2\left(\xi_1 + \frac{16}{\pi^2}\right) + \alpha_2^2\left(\xi_2 + \frac{4}{3\pi}\right)\right]. \tag{C.14}$$

For $\alpha_1 = 0$, $\alpha_2 = 0$ one recovers the original non-extremal black three-brane metric

$$ds^2 = (2\pi T)^2(1-y^2)^{-1/2}\left(-y^2dt^2 + d\boldsymbol{x}^2\right) + \frac{dy^2}{(1-y^2)^2}\,. \tag{C.15}$$

Expanding the jacobian to the leading order in α_1, α_2 we find

$$\begin{aligned} \frac{\partial y}{\partial r} = (1-y^2)\Bigg[1 &+ \alpha_1^2\left(2\rho_1^2 + 2(1-y^2)^2(\rho_1')^2 - \frac{1-y^4}{y}A_1'\right) + \\ &+ \alpha_2^2\left(\frac{1}{2}\chi_2^2 + \frac{2}{3}(1-y^2)^2(\chi_2')^2 - \frac{1-y^4}{y}A_2'\right)\Bigg]. \end{aligned} \tag{C.16}$$

Correspondingly, the shear mode damping constant (C.5) can be written as a series expansion

$$\mathcal{D} = \frac{1}{2\pi T}\int_0^1 dy y\left(1 + \alpha_1^2F_1(y) + \alpha_2^2F_2(y) + \cdots\right), \tag{C.17}$$

where ellipses stand for terms of higher order in α_1, α_2, and

$$F_1 = \frac{1-y^4}{y}A_1' - 4\bar{A}_1 - 2\rho_1^2 - 2(1-y^2)^2(\rho_1')^2 + \frac{16}{\pi^2}\,, \tag{C.18}$$

$$F_2 = \frac{1-y^4}{y}A_2' - 4\bar{A}_2 - \frac{1}{2}\chi_2^2 - \frac{2}{3}(1-y^2)^2(\chi_2')^2 + \frac{4}{3\pi}\,, \tag{C.19}$$

where $\bar{A}_1 = A_1(\xi_1 = 0)$, $\bar{A}_1 = A_2(\xi_2 = 0)$. Using equations of motion (C.7)–(C.8), (C.9)–(C.10) we now show that F_1 and F_2 vanish identically.

To prove that $F_1 \equiv 0$, first we integrate eq. (C.9) from 0 to y (taking into account that $\bar{A}_1(0) = 0$, $A_1'(0) = 0$) to obtain the identity

$$\bar{A}_1(y) = \frac{1}{2}y(1-y^2)A_1' + 2\int_0^y z(1-z^2)(\rho_1')^2 dz\,. \tag{C.20}$$

Inserting this into eq. (C.18) we find

$$F_1 = 1 - \rho_1^2 - (1-y^2)^2(\rho_1')^2 - 2\int_0^y \frac{1-z^4}{z}(\rho_1')^2 dz\,. \tag{C.21}$$

To see that the right hand side of eq. (C.21) is identically zero, we multiply both sides of the differential equation (C.7) by ρ_1'. We have then

$$(\rho_1')^2 = -\frac{y(\rho_1^2)'}{(1-y^2)^2} - \frac{y}{2}[(\rho_1')^2]'\,. \tag{C.22}$$

This allows us to express the integrand in eq. (C.21) as

$$\begin{aligned}\frac{2(1-z^4)}{z}(\rho_1')^2 &= -\frac{1+z^2}{1-z^2}(\rho_1^2)' - (1-z^4)[(\rho_1')^2]' \\ &= -(\rho_1^2)' - (1-z^2)^2[(\rho_1')^2]' + 4z(1-z^2)(\rho_1')^2 - \\ &\quad -\frac{2z^2}{1-z^2}(\rho_1^2)' - 2z^2(1-z^2)[(\rho_1')^2]' - 4z(1-z^2)(\rho_1')^2\,.\end{aligned} \tag{C.23}$$

Eq. (C.22) ensures that the last line in eq. (C.23) adds up to zero. Noting that

$$-(\rho_1^2)' - (1-z^2)^2[(\rho_1')^2]' + 4z(1-z^2)(\rho_1')^2 = \frac{d}{dz}\left(-\rho_1^2 - (1-z^2)^2(\rho_1')^2\right) \tag{C.24}$$

and remembering that $\rho_1(0) = 1$, $\rho_1'(0) = 0$ (see (C.11)), we arrive at

$$2\int_0^y \frac{1-z^4}{z}(\rho_1')^2 dz = 1 - \rho_1^2 - (1-y^2)^2(\rho_1')^2\,. \tag{C.25}$$

Consequently, $F_1 \equiv 0$. The proof of $F_2 \equiv 0$ is similar. We conclude therefore that (at least) to the next to leading order in the high temperature expansion parameters m_b/T, m_f/T the shear mode damping constant of the $\mathcal{N} = 2^*$ gauge theory at large-N and large 't Hooft coupling is independent of the parameters of deformation (hypermultiplet masses), and equals $1/4\pi T$.

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